Int. J. Solids Structures Vol. 20, No. 4, pp. 333–350, 1984
Printed in Great Britain.

0020–7683/84 $3.00 + .00
Pergamon Press Ltd.

A NEW MODEL FOR THIN PLATES WITH RAPIDLY VARYING THICKNESS†

ROBERT V. KOHN‡ and MICHAEL VOGELIUS§
Courant Institute of Mathematical Sciences, 251 Mercer St., New York, NY 10012, U.S.A.

(*Received* 21 *February* 1983; *in revised form* 18 *July* 1983)

Abstract—We study the bending of a thin plate with rapidly varying thickness, for example one with rib-like stiffeners or perforated by small holes. We obtain a fourth-order equation for the midplane displacement, using an asymptotic analysis based on three-dimensional linear elasticity. The coefficients of this equation represent the constitutive law relating bending moments to midplane curvature; they are explicitly determined by the plate geometry. Our analysis distinguishes between three different cases, in which the thickness varies on a length scale longer than, on the order of, or shorter than the mean thickness.

1. INTRODUCTION

We study pure bending of a linearly elastic plate with rapidly varying thickness. We restrict our attention to symmetric plates, and to loads transverse to the midplane. Our method is that of asymptotic analysis, using a multiple-scale approach[6, 28, 41] as the plate thickness tends to zero. The fine-scale structure of the thickness variation is assumed to be periodic or quasiperiodic; the model encompasses, for example, plates stiffened by one or more families of ribs, or perforated by finely spaced holes.

Let ϵ denote the mean plate thickness; we suppose that the thickness varies with length scale ϵ^a, $0 < a < \infty$, and that the load per unit mid-plane area is $\epsilon^3 F(x_1, x_2)$. As $\epsilon \to 0$, we find that

(a) The limiting mid-plane displacement $w(x_1, x_2)$ satisfies a fourth order equation

$$\sum_{\alpha,\beta,\gamma,\delta=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(M_{\alpha\beta\gamma\delta}\frac{\partial^2 w}{\partial x_\gamma \partial x_\delta}\right) = F. \tag{1.1}$$

(b) The tensor $M_{\alpha\beta\gamma\delta}(x_1, x_2)$ depends explicitly on the plate's geometry, through certain auxiliary functions which may be computed numerically.

(c) One can estimate the stress in the three-dimensional plate, with error $o(\epsilon)$ as $\epsilon \to 0$. *The formulas for* $M_{\alpha\beta\gamma\delta}$ *and for the stress in the three-dimensional plate depend critically on whether* $a < 1$, $a = 1$, *or* $a > 1$. In a sense, therefore, we are presenting not one model but three separate ones, for geometry variations slower than, on the order of, and faster than the mean thickness. When $a < 1$, our model is equivalent to "homogenizing the Kirchhoff plate equation"; when $a > 1$, it corresponds to "homogenizing the rough boundary, then using Kirchhoff plate theory". The case $a = 1$ apparently has no such simple interpretation.

Plates with densely spaced stiffeners have recently attracted much attention in the literature on structural optimization[11–13, 31, 32, 39]. One expects a plate with properly designed stiffeners to be more efficient in the use of material—i.e. stronger per unit weight—than any uniform or slowly-varying structure, in certain design contexts. Several authors have demonstrated this assertion, using specific models for the behavior of stiffened plates[11, 12, 25, 33]. These developments point up the need for a systematic study

†The numerical work reported in Section 7 was supported by the Computer Science Center of the University of Maryland, U.S.A.
‡Supported in part by NSF research grant MCS-8201308.
§Supported in part by NSF research grant MCS 78-02920. Current address: Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, U.S.A.

of variable-thickness structures, based on three-dimensional linear elasticity; the present work represents a step toward that goal.

Many authors have used asymptotic analysis to derive plate and shell theories, see e.g.[14, 19–21, 26, 40], and it has been recognized that rapid thickness variation can be allowed if the expansion is done properly[26]. The homogenization of plate equations, on the other hand, has also been studied extensively[2, 3, 12, 13, 16, 17, 30, 32, 34, 35, 39]. The methods of this paper are a blend of these two approaches. Our work is closely related to that of Caillerie[8–10], who has studied flat plates with periodically varying composition, using a scaling that corresponds to our $a = 1$.

The analysis presented here is purely formal. We feel confident, however, that our model represents the correct limiting behavior of the three-dimensional solution, under modest regularity hypotheses on $h(\mathbf{x}; \eta)$. A convergence result of this type is proved in[23] for the $a = 1$ scaling, in case $h = h(\eta)$ is purely periodic and the plate edges are clamped. The corresponding assertion for a homogeneous, flat plate is, of course, well known[4, 36–38, 42], and an analogous result is proved in[10] for the problem studied there.

We wish to thank George Papanicolaou for sharing generously both his knowledge and his enthusiasm in many discussions related to this work.

2. PRELIMINARIES

This section establishes notation, describes precisely the problem we consider, and summarizes briefly our method.

We shall write $\underset{\sim}{x} = (x_1, x_2, x_3)$ for vectors in $\mathbb{R}^3$, and $\mathbf{x} = (x_1, x_2)$ for vectors in $\mathbb{R}^2$. Latin indices will range from 1 to 3, and Greek ones from 1 to 2; the summation convention applies whenever indices are repeated. We write $\partial_i = \partial/\partial x_i$ and $\partial_{ij} = \partial^2/\partial x_i \partial x_j$.

(a) *Linear elasticity*

Associated with any displacement $\underset{\sim}{u} = (u_1, u_2, u_3)$ of $\mathbb{R}^3$ is its strain tensor

$$e_{ij}(\underset{\sim}{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{2.1}$$

and the corresponding stress tensor

$$\sigma_{ij}(\underset{\sim}{u}) = B_{ijkl} e_{kl}(\underset{\sim}{u}). \tag{2.2}$$

The fourth-order tensor B_{ijkl} satisfies

$$B_{ijkl} = B_{jikl} = B_{ijlk} = B_{klij}; \tag{2.3}$$

we assume the elastic energy

$$B_{ijkl} e_{ij} e_{kl} \tag{2.4}$$

is positive definite on symmetric tensors.

We shall always assume that the horizontal planes are planes of elastic symmetry. This means[29]

$$B_{\alpha\beta\gamma 3} = 0, \quad B_{\alpha 333} = 0, \tag{2.5}$$

so that Hooke's law (2.2) becomes

$$\begin{aligned} \sigma_{\alpha\beta} &= B_{\alpha\beta\gamma\delta} e_{\gamma\delta} + B_{\alpha\beta 33} e_{33} \\ \sigma_{\alpha 3} &= 2B_{\alpha 3\beta 3} e_{\beta 3} \\ \sigma_{33} &= B_{33\alpha\beta} e_{\alpha\beta} + B_{3333} e_{33}. \end{aligned} \tag{2.6}$$

We define

$$\tilde{B}_{\alpha\beta\gamma\delta} = B_{\alpha\beta\gamma\delta} - \frac{B_{\alpha\beta33}B_{\gamma\delta33}}{B_{3333}}; \tag{2.7}$$

according to standard Kirchhoff plate theory, the rigidity tensor associated to a flat plate with (rescaled) thickness $2h$ is

$$M_{\alpha\beta\gamma\delta} = \frac{2}{3}h^3\tilde{B}_{\alpha\beta\gamma\delta}.$$

(See Section 2e).

For an isotropic material with Young's modulus E and Poisson's ratio ν, the nonzero components of B_{ijkl} are

$$\begin{aligned} B_{iiii} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \\ B_{iijj} &= \frac{E\nu}{(1+\nu)(1-2\nu)} \quad i \neq j \\ B_{ijij} = B_{ijji} &= \frac{1}{2}\frac{E}{1+\nu} \quad i \neq j \end{aligned} \tag{2.8}$$

(no summation convention). In that case

$$\begin{aligned} \tilde{B}_{1111} = \tilde{B}_{2222} &= \frac{E}{(1-\nu^2)} \\ \tilde{B}_{1212} = \tilde{B}_{1221} = \tilde{B}_{2112} = \tilde{B}_{2121} &= \frac{1}{2}\frac{E}{1+\nu} \\ \tilde{B}_{1122} = \tilde{B}_{2211} &= \frac{E\nu}{(1-\nu^2)}. \end{aligned} \tag{2.9}$$

(b) *Plate geometry*

The plate geometry is determined by

a domain Ω in the x_1–x_2 plane, representing the midplane; (2.10a)

a real parameter a, $0 < a < \infty$, determining the length scale of the thickness variation; and (2.10b)

a function $h(\mathbf{x}; \boldsymbol{\eta}) \geq 0$, defined for $\mathbf{x} \in \bar{\Omega}$ and $\boldsymbol{\eta} \in \mathbb{R}^2$. (2.10c)

The three-dimensional region occupied by the plate is

$$R(\epsilon) = \{\underset{\sim}{x} : \mathbf{x} \in \Omega, |x_3| < \epsilon h(\mathbf{x}; \mathbf{x}/\epsilon^a)\}, \tag{2.11}$$

where ϵ is a small parameter.

We allow $h(\mathbf{x}; \boldsymbol{\eta}) = 0$ for some values of $(\mathbf{x}; \boldsymbol{\eta})$; hence the three-dimensional plate may have holes. We also allow h to be discontinuous. We assume, however, that

For each $\mathbf{x} \in \bar{\Omega}$, $\{\boldsymbol{\eta} \in \mathbb{R}^2 : h(\mathbf{x}; \boldsymbol{\eta}) > 0\}$ is connected; and (2.12a)

h is bounded away from zero on the set where it does not vanish, i.e. $h(\mathbf{x}; \boldsymbol{\eta}) > 0 \Rightarrow h(\mathbf{x}; \boldsymbol{\eta}) > c$, for some constant $c > 0$. (2.12b)

A further hypothesis is needed concerning the dependence of h on $\boldsymbol{\eta}$. We shall focus in Sections 3–5 on the simplest case, that of locally periodic structure:

$$h(\mathbf{x};\boldsymbol{\eta}) \text{ is periodic with period 1 in } \eta_1 \text{ and } \eta_2. \tag{2.13a}$$

The case of "quasiperiodic" local structure will be addressed in Section 6; by this we mean:

$$\begin{aligned}&h(\mathbf{x};\boldsymbol{\eta}) = H(\mathbf{x};\boldsymbol{\alpha}_1(\mathbf{x})\cdot\boldsymbol{\eta};\ldots;\boldsymbol{\alpha}_N(\mathbf{x})\cdot\boldsymbol{\eta}), \text{ for some}\\ &\text{functions } \boldsymbol{\alpha}_i(\mathbf{x})\in\mathbb{R}^2 (1\le i\le N) \text{ and } H(\mathbf{x};t_1,\ldots,t_N)\ge 0,\\ &H \text{ being periodic in } t_1,\ldots,t_N \text{ with period 1.}\end{aligned} \tag{2.13b}$$

To clarify the scope of the latter hypothesis, consider a plate with N families of ribs, each having smoothly varying density and direction. Suppose $g_i(t)$, $|t| < 1/2$, represents the profile of a single rib in the ith family; let $|\boldsymbol{\alpha}_i(\mathbf{x})|$ represent the relative density of the ith family at $\mathbf{x}$, and when $\boldsymbol{\alpha}_i \neq 0$ let $\boldsymbol{\alpha}_i/|\boldsymbol{\alpha}_i|$ be directed orthogonal to the ribs. One can model this situation by extending $g_i(t)$ periodically and taking

$$\begin{aligned} H(t_1,\ldots,t_N) &= \max_{1\le i\le N} g_i(t_i)\\ h(\mathbf{x};\boldsymbol{\eta}) &= H(\boldsymbol{\alpha}_1\cdot\boldsymbol{\eta};\ldots;\boldsymbol{\alpha}_N\cdot\boldsymbol{\eta}).\end{aligned} \tag{2.14}$$

We shall denote by $\partial_+R(\epsilon)$ and $\partial_-R(\epsilon)$ the upper and lower faces of the plate. If $h(\mathbf{x},\boldsymbol{\eta})$ is continuous then

$$\partial_\pm R(\epsilon) = \{\underline{x} : \mathbf{x}\in\Omega,\ x_3 = \pm\epsilon h(\mathbf{x},\mathbf{x}/\epsilon^a)\}; \tag{2.15}$$

if h is discontinuous than $\partial_\pm R(\epsilon)$ has vertical parts as well.

We assume for simplicity that the plate is homogeneous. (The case of varying material composition can, however, be treated using the same method.)

It will often be necessary to average a periodic function $g(\mathbf{x};\boldsymbol{\eta})$ with respect to $\boldsymbol{\eta}$:

$$\mathcal{M}g(\mathbf{x}) = \frac{1}{p_1p_2}\int_0^{p_1}\int_0^{p_2} g(\mathbf{x};\boldsymbol{\eta})\,d\eta_1\,d\eta_2, \tag{2.16}$$

where p_i are the periods of $g(\mathbf{x};\cdot)$. In what follows, all functions will have period 1 unless explicitly stated otherwise.

(c) *Loads and equations of equilibrium*

We suppose that the plate is loaded along its faces $\partial_\pm R(\epsilon)$ by forces $\epsilon^3(0,0,f_\pm(\mathbf{x},\mathbf{x}/\epsilon^a))$ *per unit midplane area*, where

$$\begin{aligned}&f_+(\mathbf{x};\boldsymbol{\eta}) \text{ and } f_-(\mathbf{x};\boldsymbol{\eta}) \text{ are defined for } \mathbf{x}\in\Omega \text{ and } \boldsymbol{\eta}\in\mathbb{R}^2,\\ &\text{and are periodic or quasiperiodic in } \boldsymbol{\eta}\text{; also,}\\ &f_+(\mathbf{x};\boldsymbol{\eta}) = f_-(\mathbf{x};\boldsymbol{\eta}) = 0 \text{ whenever } h(\mathbf{x};\boldsymbol{\eta}) = 0.\end{aligned} \tag{2.17}$$

The scaling of the load has been chosen so that the vertical displacement of the plate remains bounded as $\epsilon\to 0$. We allow the load to vary rapidly, since in practice one might load a ribbed plate just along the top of the ribs. Only the mean force per unit midplane area

$$F(x) = \mathcal{M}(f_+ + f_-) \tag{2.18}$$

will appear in the limiting equation for the midplane displacement.

The equations of elastostatic equilibrium for the ϵ-dependent three-dimensional plate are

$$\partial_j[\sigma_{ij}(\underline{u}^\epsilon)] = 0 \quad \text{in} \quad R(\epsilon), \quad i = 1, 2, 3, \tag{2.19}$$

where $\sigma_{ij}(\underline{u})$ is given by (2.2). The boundary conditions on the plate faces are

$$\sigma_{ij}^\epsilon n_j^\epsilon = \begin{cases} 0 & i = 1, 2 \\ \epsilon^3 f_\pm |n_3^\epsilon| & i = 3 \end{cases} \quad \text{on } \partial_\pm R(\epsilon), \tag{2.20}$$

where $\underline{n}^\epsilon$ is the outward unit normal vector, i.e.

$$|n_3^\epsilon|^{-1}\underline{n}^\epsilon = \left(-\epsilon\frac{\partial h}{\partial x_1} - \epsilon^{1-a}\frac{\partial h}{\partial \eta_1}, -\epsilon\frac{\partial h}{\partial x_2} - \epsilon^{1-a}\frac{\partial h}{\partial \eta_2}, \pm 1\right) \tag{2.21}$$

whenever h is differentiable. We postpone the discussion of boundary conditions at the plate edges until the end of Section 2(d).

(d) *The method*

Our goal is to find a limiting "effective equation" for the vertical displacement u_3^ϵ as $\epsilon \to 0$. We use a variational approach, with the well-known method of multiple-scale asymptotic expansions[6, 41] as motivation. The following discussion applies only to the locally periodic case (2.13a).

One begins by postulating an ansatz for $\underline{u}^\epsilon$

$$\underline{u}^\epsilon \sim \underline{u}^* = \sum_{i=0}^{k} \epsilon^{t_i}\underline{u}^{(i)}(\mathbf{x}; \mathbf{x}/\epsilon^a; x_3/\epsilon). \tag{2.22}$$

Each $\underline{u}^{(i)} = \underline{u}^{(i)}(\mathbf{x}; \boldsymbol{\eta}; \xi)$ is defined for $\mathbf{x} \in \Omega$, $\boldsymbol{\eta} \in \mathbb{R}^2$, $|\xi| < h(\mathbf{x}; \boldsymbol{\eta})$, and is periodic in η_1, η_2. The first term in (2.22) has the form

$$\epsilon^{t_0}\underline{u}^{(0)}(\mathbf{x}; \boldsymbol{\eta}; \xi) = (0, 0, w(\mathbf{x})),$$

where w, the limiting vertical displacement, is an as yet undetermined function of $\mathbf{x}$.

The remaining exponents $t_i > 0$ and functions $\underline{u}^{(i)}$ are chosen differently depending on whether $a < 1$, $a = 1$, or $a > 1$. In each case, the formulas have been obtained by substituting the formal expansion (2.22) into (2.19) and (2.20), collecting terms with like powers of ϵ, and solving successively for the functions $\underline{u}^{(i)}$. Thus (2.22) represents the first few terms of a full asymptotic expansion for $\underline{u}^\epsilon$; we keep only those terms that produce strains of order ϵ. The procedure just described is now standard; we shall therefore present only the ansatz so obtained, omitting the details of its derivation.

The fourth-order "effective equation" for w is obtained as follows. One can write the strain of $\underline{u}^*$ in the form

$$e_{ij}(\underline{u}^*) = \epsilon \mathrm{X}_{ij}^{\alpha\beta}(\mathbf{x}; \mathbf{x}/\epsilon^a; x_3/\epsilon)\partial_{\alpha\beta}w + o(\epsilon), \tag{2.23}$$

where $\mathrm{X}_{ij}^{\alpha\beta}(\mathbf{x}; \boldsymbol{\eta}; \xi)$ are explicit, $\boldsymbol{\eta}$-periodic functions depending only on the plate geometry, and

$$\mathrm{X}_{ij}^{\alpha\beta} = \mathrm{X}_{ji}^{\beta\alpha}. \tag{2.24}$$

Substituting (2.23) into the energy expression

$$\frac{1}{2}\int_{R(\epsilon)} B_{ijkl}e_{ij}(\underline{u}^*)e_{kl}(\underline{u}^*)\,\mathrm{d}\underline{x} - \epsilon^3\int_\Omega u_3^*(f_+ + f_-)\,\mathrm{d}\mathbf{x},$$

integrating with respect to ξ, averaging in η, and discarding terms that are $o(\epsilon^3)$, one is led to consider

$$\frac{1}{2}\epsilon^3\int_\Omega M_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w\partial_{\gamma\delta}w\,\mathrm{dx}-\epsilon^3\int_\Omega wF\,\mathrm{dx}, \tag{2.25}$$

where F is given by (2.18) and

$$M_{\alpha\beta\gamma\delta}(\mathrm{x})=\mathcal{M}\left(\int_{-h(\mathrm{x};\,\eta)}^{+h(\mathrm{x};\,\eta)}B_{ijkl}X_{ij}^{\alpha\beta}X_{kl}^{\gamma\delta}\,\mathrm{d}\xi\right). \tag{2.26}$$

The function w *must minimize* (2.25), i.e. it satisfies the *effective equation*

$$\partial_{\alpha\beta}(M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w)=F \text{ in } \Omega. \tag{2.27}$$

Once w is known, (2.23) gives an approximation for the strain of $\underline{u}^\epsilon$ in the three-dimensional plate, with error $o(\epsilon)$.

Note that by (2.3) and (2.24)

$$M_{\alpha\beta\gamma\delta}=M_{\beta\alpha\gamma\delta}=M_{\alpha\beta\delta\gamma}=M_{\gamma\delta\alpha\beta}; \tag{2.28}$$

also, since the elasticity tensor B_{ijkl} is positive definite,

$$M_{\alpha\beta\gamma\delta}t_{\alpha\beta}t_{\gamma\delta}\geq 0 \tag{2.29}$$

for any symmetric 2×2 tensor $t_{\alpha\beta}$, and

$$M_{\alpha\beta\gamma\delta}t_{\alpha\beta}t_{\gamma\delta}=0 \Leftrightarrow X_{ij}^{\alpha\beta}t_{\alpha\beta}=0 \quad \text{all } i,j. \tag{2.30}$$

Equation (2.27) must be combined with appropriate boundary conditions for w at $\partial\Omega$. At a clamped edge, the three-dimensional solution satisfies

$$\underline{u}^\epsilon=0 \quad \text{on } C(\epsilon), \tag{2.31}$$

where $C(\epsilon)$ denotes the plate edge

$$C(\epsilon)=\{\underline{x}:\mathrm{x}\in\partial\Omega,\ |x_3|<\epsilon h(\mathrm{x};\,\mathrm{x}/\epsilon^a)\}. \tag{2.32}$$

The boundary conditions for w are obtained by imposing (2.31) on the leading terms of (2.22):

$$w=0,\quad n_\alpha\partial_\alpha w=0 \quad \text{on } \partial\Omega. \tag{2.33}$$

At a simply-supported edge, $\underline{u}^\epsilon$ satisfies one displacement boundary condition

$$u_3^\epsilon=0 \quad \text{on } C(\epsilon), \tag{2.34a}$$

and the complementing "natural" conditions

$$\sigma_{\alpha\beta}(\underline{u}^\epsilon)n_\beta=0 \quad \text{on } C(\epsilon). \tag{2.34b}$$

The corresponding conditions for w are the analogue of (2.34a),

$$w=0 \quad \text{on } \partial\Omega \tag{2.35a}$$

and the complementing "natural" boundary condition for the variational problem (2.25),

$$M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w\cdot n_\alpha\cdot n_\beta=0 \quad \text{on } \partial\Omega. \tag{2.35b}$$

Similarly, at a free edge w satisfies the full set of "natural" boundary conditions for (2.25),

$$M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w \cdot n_\alpha \cdot n_\beta = 0 \tag{2.36a}$$

$$\partial_\alpha(M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w)n_\beta + \tau_\nu\partial_\nu(M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w \cdot n_\alpha \cdot \tau_\beta) = 0. \tag{2.36b}$$

The vector $\mathbf{n} = (n_1, n_2)$ in (2.33)–(2.36) is the unit vector normal to $\partial\Omega$, and $\boldsymbol{\tau}$ in (2.36b) is the unit tangent vector along $\partial\Omega$, counterclockwise.

2(e) *Slowly-varying plates*

To clarify the method, we sketch briefly how it applies when the plate thickness varies slowly—i.e. when $h = h(x)$ does not depend on $\boldsymbol{\eta}$. One recovers Kirchhoff plate theory[1, 38] in this case.

The appropriate form for the ansatz (2.22) is

$$\underline{u}^* = \left(-x_3\partial_1 w, -x_3\partial_2 w, w + \frac{1}{2}\frac{B_{\alpha\beta33}}{B_{3333}}(x_3)^2\partial_{\alpha\beta}w\right). \tag{2.37}$$

One computes that (2.23) holds with

$$\mathrm{X}_{ij}^{\alpha\beta}(x;\xi) = -\xi\delta_{ij}^{\alpha\beta} + \xi\frac{B_{\alpha\beta33}}{B_{3333}}\delta_{ij}^{33},$$

using the notation

$$\delta_{kl}^{ij} = \begin{cases} 1 & \text{if } (i,j) = (k,l) \text{ as ordered pairs} \\ 0 & \text{otherwise.} \end{cases} \tag{2.38}$$

It follows that

$$M_{\alpha\beta\gamma\delta} = \frac{2}{3}h^3(\mathbf{x}) \cdot \tilde{B}_{\alpha\beta\gamma\delta}, \tag{2.39}$$

where $\tilde{B}_{\alpha\beta\gamma\delta}$ is defined by (2.7).

For the isotropic constitutive law (2.8), substitution of (2.9) into (2.39) leads to the effective equation

$$\partial_{\alpha\beta}\left[D\left(\partial_{\alpha\beta}w + \frac{\nu}{1-\nu}\delta_\beta{}^\alpha\Delta w\right)\right] = F$$

where

$$D = \frac{2E}{3(1+\nu)}h^3(\mathbf{x}).$$

3. THE CASE $a < 1$

Suppose (2.13a) holds, i.e. h is periodic in $\boldsymbol{\eta}$, and $a < 1$. The mean thickness of the plate is then much smaller than the length scale of the thickness variation.

For the duration of this section only we define

$$Q(\mathbf{x}) = \{\boldsymbol{\eta} \in \mathbb{R}^2 : h(\mathbf{x};\boldsymbol{\eta}) > 0\}. \tag{3.1}$$

The ansatz for $a < 1$ depends on auxiliary $\boldsymbol{\eta}$-periodic functions $\phi^{\alpha\beta}(\mathbf{x};\boldsymbol{\eta})$ ($\alpha, \beta = 1, 2$; $\phi^{12} = \phi^{21}$). They satisfy

$$\frac{\partial^2}{\partial\eta_\gamma\partial\eta_\delta}\left(h^3\tilde{B}_{\gamma\delta\rho\sigma}\frac{\partial^2\phi^{\alpha\beta}}{\partial\eta_\rho\partial\eta_\sigma}\right) = \frac{-\partial^2}{\partial\eta_\gamma\partial\eta_\delta}(h^3\tilde{B}_{\gamma\delta\alpha\beta}) \tag{3.2}$$

in $Q(\mathbf{x})$. If $\partial Q(\mathbf{x}) \neq \emptyset$, the boundary condition there is the natural one corresponding to the minimization

$$\min_{\phi} \mathscr{M}\left[h^3 \tilde{B}_{\gamma\delta\rho\sigma} \frac{\partial^2 \phi}{\partial \eta_\gamma \partial \eta_\sigma} \frac{\partial^2}{\partial \eta_\rho \partial \eta_\sigma} (\phi + \eta_\alpha \eta_\beta) \right], \tag{3.3}$$

the minimum being taken among $\boldsymbol{\eta}$-periodic functions. One verifies easily that $\phi^{\alpha\beta}$ exists and is unique on $Q(\mathbf{x})$, up to an additive function of $\mathbf{x}$. Note that $\mathbf{x}$ enters (3.2) and (3.3) only as a parameter.

The ansatz (2.22) for this case is

$$\begin{aligned} u_\gamma^* &= -x_3 \partial_\gamma w - \epsilon^a x_3 \frac{\partial}{\partial \eta_\gamma} (\phi^{\alpha\beta}) \partial_{\alpha\beta} w - \epsilon^{2a} x_3 \partial_\gamma (\phi^{\alpha\beta} \partial_{\alpha\beta} w) \\ u_3^* &= w + \epsilon^{2a} \phi^{\alpha\beta} \partial_{\alpha\beta} w + \frac{1}{2} (x_3)^2 \frac{B_{33\gamma\delta}}{B_{3333}} \frac{\partial^2}{\partial \eta_\gamma \partial \eta_\delta} \left(\frac{1}{2} \eta_\alpha \eta_\beta + \phi^{\alpha\beta} \right) \partial_{\alpha\beta} w. \end{aligned} \tag{3.4}$$

Recall that $\partial_\gamma = \partial / \partial x_\gamma$; the right side of (3.4) must be evaluated at $\boldsymbol{\eta} = \mathbf{x}/\epsilon^a$ *after* differentiation. One computes that (2.23) holds with

$$\begin{aligned} X_{\gamma\delta}^{\alpha\beta} &= -\xi \frac{\partial^2}{\partial \eta_\gamma \partial \eta_\delta} \left(\frac{1}{2} \eta_\alpha \eta_\beta + \phi^{\alpha\beta} \right) \\ X_{3\gamma}^{\alpha\beta} &= 0 \\ X_{33}^{\alpha\beta} &= \xi \frac{B_{33\gamma\delta}}{B_{3333}} \frac{\partial^2}{\partial \eta_\gamma \partial \eta_\delta} \left(\frac{1}{2} \eta_\alpha \eta_\beta + \phi^{\alpha\beta} \right). \end{aligned} \tag{3.5}$$

Substituting (3.5) into (2.26), we obtain

$$M_{\alpha\beta\gamma\delta}(\mathbf{x}) = \mathscr{M}\left[\frac{2}{3} h^3 \tilde{B}_{\lambda\mu\rho\sigma} \frac{\partial^2}{\partial \eta_\lambda \partial \eta_\mu} \left(\frac{1}{2} \eta_\alpha \eta_\beta + \phi^{\alpha\beta} \right) \frac{\partial^2}{\partial \eta_\rho \partial \eta_\sigma} \left(\frac{1}{2} \eta_\gamma \eta_\delta + \phi^{\gamma\delta} \right) \right]. \tag{3.6}$$

By Green's formula, (2.16), and (3.2),

$$\mathscr{M}\left[h^3 \tilde{B}_{\lambda\mu\rho\sigma} \frac{\partial^2 \phi^{\alpha\beta}}{\partial \eta_\lambda \partial \eta_\mu} \frac{\partial^2 \phi^{\gamma\delta}}{\partial \eta_\rho \partial \eta_\sigma} \right] = -\mathscr{M}\left[h^3 \tilde{B}_{\alpha\beta\rho\sigma} \frac{\partial^2 \phi^{\gamma\delta}}{\partial \eta_\rho \partial \eta_\sigma} \right];$$

thus (3.6) may be rewritten

$$M_{\alpha\beta\gamma\delta}(\mathbf{x}) = \mathscr{M}\left[\frac{2}{3} h^3 \tilde{B}_{\alpha\beta\gamma\delta} \right] + \mathscr{M}\left[\frac{2}{3} h^3 \tilde{B}_{\alpha\beta\rho\sigma} \frac{\partial^2 \phi^{\gamma\delta}}{\partial \eta_\rho \partial \eta_\sigma} \right]. \tag{3.7}$$

If $h(\mathbf{x}; \boldsymbol{\eta})$ does not depend on $\boldsymbol{\eta}$, then $\phi^{\alpha\beta}$ is independent of $\boldsymbol{\eta}$, and (3.7) is identical with (2.39).

Hypothesis (2.12) implies that $M_{\alpha\beta\gamma\delta}$ is positive definite. Indeed, if $t_{\alpha\beta}$ is a symmetric 2×2 tensor and $M_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} = 0$, then $X_{\gamma\delta}^{\alpha\beta} t_{\alpha\beta} = 0$ by (2.30), whence by (3.5)

$$\frac{\partial^2}{\partial \eta_\gamma \partial \eta_\delta} (\phi^{\alpha\beta} t_{\alpha\beta}) = -t_{\gamma\delta}. \tag{3.8}$$

Since $\phi^{\alpha\beta}$ is $\boldsymbol{\eta}$-periodic, (2.12) and (3.8) imply $t_{\gamma\delta} = 0$ ($\gamma, \delta = 1, 2$).

The same effective equation (2.27), (3.7) is also obtained if one homogenizes the standard plate equation (2.27), (2.39), see[17].

4. THE CASE $a = 1$

Suppose h is periodic in $\boldsymbol{\eta}$ and $a = 1$. In this case the thickness varies on the same scale as the mean thickness.

The following conventions apply to this section only. We shall write η_3 instead of ξ for the rescaled vertical coordinate x_3/ϵ. If $\underline{\phi} = (\phi_1, \phi_2, \phi_3)$ is a function of $\underline{\eta} = (\eta_1, \eta_2, \eta_3)$ we define

$$E_{ij}(\underline{\phi}) = \frac{1}{2}\left(\frac{\partial \phi_i}{\partial \eta_j} + \frac{\partial \phi_j}{\partial \eta_i}\right)$$
$$\Sigma_{ij}(\underline{\phi}) = B_{ijkl}E_{kl}(\underline{\phi}). \tag{4.1}$$

We denote by $Q(\mathbf{x})$ the rescaled periodically-varying "plate" with thickness $h(\mathbf{x}; \cdot)$

$$Q(\mathbf{x}) = \{\underline{\eta} : |\eta_3| < h(\mathbf{x}; \boldsymbol{\eta})\}; \tag{4.2}$$

$\partial_+ Q(\mathbf{x})$ and $\partial_- Q(\mathbf{x})$ represent its upper and lower faces, i.e. (when h is continuous)

$$\partial_\pm Q(\mathbf{x}) = \{\underline{\eta} : \eta_3 = \pm h(\mathbf{x}; \boldsymbol{\eta})\};$$

and $\underline{\nu}(\mathbf{x}; \boldsymbol{\eta})$ is the outward unit vector normal to $\partial_\pm Q(\mathbf{x})$.

The ansatz for $a = 1$ depends on auxiliary functions $\underline{\phi}^{\alpha\beta}(\mathbf{x}; \underline{\eta})$ $(\alpha, \beta = 1, 2; \underline{\phi}^{12} = \underline{\phi}^{21})$ which are periodic in η_1 and η_2. They solve

$$\frac{\partial}{\partial \eta_j}\Sigma_{ij}(\underline{\phi}^{\alpha\beta}) = 0 \quad \text{in } Q(\mathbf{x}), \tag{4.3}$$

with boundary conditions

$$\Sigma_{ij}\nu_j = \begin{cases} \eta_3 \tilde{B}_{ij\alpha\beta}\nu_j & i = 1, 2 \\ 0 & i = 3 \end{cases} \tag{4.4}$$

on $\partial_\pm Q(\mathbf{x})$. One verifies by a standard variational argument that $\underline{\phi}^{\alpha\beta}$ exists and is unique up to an additive function of $\mathbf{x}$. Once again, $\mathbf{x}$ enters (4.3), (4.4) only as a parameter determining the geometry of $Q(\mathbf{x})$.

The ansatz (2.22) for this case is

$$\underline{u}^* = \left(-x_3\partial_1 w, -x_3\partial_2 w, w + \frac{1}{2}\frac{B_{\alpha\beta 33}}{B_{3333}}(x_3)^2\partial_{\alpha\beta}w\right) + \epsilon^2\underline{\phi}^{\alpha\beta}(\mathbf{x}; \underline{x}/\epsilon)\partial_{\alpha\beta}w. \tag{4.5}$$

One computes that (2.23) holds with

$$\mathrm{X}^{\alpha\beta}_{\lambda\mu} = -\eta_3\delta^{\alpha\beta}_{\lambda\mu} + E_{\lambda\mu}(\underline{\phi}^{\alpha\beta})$$
$$\mathrm{X}^{\alpha\beta}_{\lambda 3} = E_{\lambda 3}(\underline{\phi}^{\alpha\beta}) \tag{4.6}$$
$$\mathrm{X}^{\alpha\beta}_{33} = \eta_3\frac{B_{\alpha\beta 33}}{B_{3333}} + E_{33}(\underline{\phi}^{\alpha\beta}),$$

$\delta^{\alpha\beta}_{\lambda\mu}$ being defined by (2.38). Substituting (4.6) into (2.26), we obtain

$$M_{\alpha\beta\gamma\delta} = \mathcal{M}\left[\int (\eta_3)^2\tilde{B}_{\alpha\beta\gamma\delta}\,\mathrm{d}\eta_3\right] - \mathcal{M}\left[\int \eta_3\tilde{B}_{\lambda\mu\gamma\delta}E_{\lambda\mu}(\underline{\phi}^{\alpha\beta})\,\mathrm{d}\eta_3\right]$$
$$- \mathcal{M}\left[\int \eta_3\tilde{B}_{\alpha\beta\lambda\mu}E_{\lambda\mu}(\underline{\phi}^{\gamma\delta})\,\mathrm{d}\eta_3\right] + \mathcal{M}\left[\int E_{ij}(\underline{\phi}^{\alpha\beta})\Sigma_{ij}(\underline{\phi}^{\gamma\delta})\,\mathrm{d}\eta_3\right] \tag{4.7}$$

each integral being taken over the interval $|\eta_3| < h(\mathbf{x}; \boldsymbol{\eta})$. By Green's formula, (4.3)–(4.4), and (2.16),

$$\mathcal{M}\left[\int E_{ij}(\underline{\phi}^{\alpha\beta})\Sigma_{ij}(\underline{\phi}^{\gamma\delta})\,\mathrm{d}\eta_3\right] = \mathcal{M}\left[\int \eta_3\tilde{B}_{\lambda\mu\gamma\delta}E_{\lambda\mu}(\underline{\phi}^{\alpha\beta})\,\mathrm{d}\eta_3\right], \tag{4.8}$$

therefore (4.7) may also be written

$$M_{\alpha\beta\gamma\delta} = \mathscr{M}\left[\frac{2}{3}h^3\tilde{B}_{\alpha\beta\gamma\delta}\right] - \mathscr{M}\left[\int \eta_3 \tilde{B}_{\alpha\beta\lambda\mu}E_{\lambda\mu}(\underline{\phi}^{\gamma\delta})\ \gamma_3\right]. \tag{4.9}$$

If $h(\mathbf{x};\mathbf{y})$ is independent of $\boldsymbol{\eta}$, then $\underline{\phi}^{\gamma\delta}$ is independent of $\underline{\eta}$, and coincides with (2.39).

Hypothesis (2.12) assures once again that $M_{\alpha\beta\gamma\delta}(\mathbf{x})$ is positive definite: for any 2×2 symmetric tensor $t_{\alpha\beta}$, we know from (2.30) that $M_{\alpha\beta\gamma\delta}t_{\alpha\beta}t_{\gamma\delta}=0$ implies $X^{\alpha\beta}_{\gamma\delta}t_{\alpha\beta}=0$ $(\gamma,\delta=1,2)$; in that case

$$E_{\gamma\delta}(\underline{\phi}^{\alpha\beta}t_{\alpha\beta}) = t_{\gamma\delta}\eta_3. \tag{4.10}$$

By (2.12), the slice $Q(\mathbf{x})\cap\{\eta_3=c\}$ is connected for $c\approx 0$; and by (4.10) $\boldsymbol{\phi}^{\alpha\beta}t_{\alpha\beta}$ is a periodic function with constant strain on each slice. It follows that

$$E_{\gamma\delta}(\underline{\phi}^{\alpha\beta}t_{\alpha\beta}) = 0 \quad \gamma,\delta = 1,2$$

for $|\eta_3|$ sufficiently small, whence using (4.10) $t_{\gamma\delta}=0$.

5. THE CASE $a>1$

Suppose h is periodic in $\boldsymbol{\eta}$ and $a>1$. In this case the plate thickness varies on a scale much smaller than the mean thickness.

The following notation will be used in this section and in the appendix. If $\boldsymbol{\phi}=(\phi_1,\phi_2)$ is a function of $\boldsymbol{\eta}=(\eta_1,\eta_2)$, we define

$$\begin{aligned} E_{\alpha\beta}(\boldsymbol{\phi}) &= \frac{1}{2}\left(\frac{\partial\phi_\alpha}{\partial\eta_\beta}+\frac{\partial\phi_\beta}{\partial\eta_\alpha}\right)\\ \Sigma_{\alpha\beta}(\boldsymbol{\phi}) &= B_{\alpha\beta\gamma\delta}E_{\gamma\delta}(\boldsymbol{\phi}). \end{aligned} \tag{5.1}$$

We denote by $Q(\mathbf{x};\xi)$ the rescaled slice of the periodic "plate" at height $x_3=\epsilon\xi$:

$$Q(\mathbf{x};\xi) = \{\boldsymbol{\eta} : |\xi| < h(\mathbf{x};\boldsymbol{\eta})\}; \tag{5.2}$$

$\boldsymbol{v}$ will denote the outward unit vector normal to ∂Q in the (η_1,η_2) plane. Let

$$l_{Q(\mathbf{x},\xi)}(\boldsymbol{\eta}) = \begin{cases}1 & \text{if } \boldsymbol{\eta}\in Q(\mathbf{x},\xi)\\ 0 & \text{otherwise}\end{cases} \tag{5.3}$$

and let θ be area fraction of the slice

$$\theta(\mathbf{x};\xi) = \mathscr{M}(l_{Q(\mathbf{x};\xi)}). \tag{5.4}$$

If $g(\mathbf{x};\boldsymbol{\eta};\xi)$ is defined for $\boldsymbol{\eta}\in Q(\mathbf{x},\xi)$ and is $\boldsymbol{\eta}$-periodic, we denote its average over a slice by

$$\bar{g}(\mathbf{x};\xi) = \theta(\mathbf{x};\xi)^{-1}\mathscr{M}(l_{Q(\mathbf{x};\xi)}\cdot g), \tag{5.5}$$

provided that $Q(\mathbf{x};\xi)\neq\phi$ ($l_{Q(\mathbf{x};\xi)}\cdot g$ takes the same value as g when $\boldsymbol{\eta}\in Q(\mathbf{x};\xi)$, and takes the value 0 otherwise).

The ansatz for $a>1$ depends on auxiliary functions $\boldsymbol{\psi}^{ij}(\mathbf{x};\boldsymbol{\eta};\xi)$ $(l\le i,j\le 3, \boldsymbol{\psi}^{ij}=\boldsymbol{\psi}^{ji})$. They are periodic in $\boldsymbol{\eta}$, defined for $\boldsymbol{\eta}\in Q(\mathbf{x};\xi)$, $|\xi|<\max_{\boldsymbol{\eta}} h(\mathbf{x};\boldsymbol{\eta})$, and they satisfy

$$\frac{\partial}{\partial\eta_\beta}[\Sigma_{\alpha\beta}(\boldsymbol{\psi}^{ij})] = 0 \quad \text{in } Q(\mathbf{x};\xi) \tag{5.6}$$

with boundary conditions

$$\Sigma_{\alpha\beta}(\psi^{ij})\nu_\beta = -B_{\alpha\beta ij}\nu_\beta \quad \text{on } \partial Q(\mathbf{x};\xi). \tag{5.7}$$

By (2.5), $\psi^{13} = \psi^{23} = 0$. A standard variational argument shows that ψ^{ij} exists and is unique up to a rigid motion in η. If $Q(\mathbf{x};\xi)$ is connected this rigid motion is a function of $\mathbf{x}$ and ξ only, but if $Q(\mathbf{x},\xi)$ is disconnected it may include a linear function of η, possibly different on each connected periodic component. The strain $E_{\alpha\beta}(\psi^{ij})$ is, however, unique.

The ansatz (2.22) for this case is

$$\underline{u}^* = (-x_3\partial_1 w, -x_3\partial_2 w, w) + \epsilon^2(0,0,g^{\alpha\beta})\partial_{\alpha\beta}w + \epsilon^{1+a}(\phi_1{}^{\alpha\beta}, \phi_2{}^{\alpha\beta}, 0)\partial_{\alpha\beta}w, \tag{5.8}$$

where $g^{\alpha\beta}(\mathbf{x};\xi)$ and $\phi^{\alpha\beta}(\mathbf{x};\eta;\xi)$ are defined by

$$g^{\alpha\beta}(\mathbf{x};\xi) = \int_0^\xi \tau \frac{B_{33\alpha\beta} + B_{33\rho\sigma}\overline{E_{\rho\sigma}(\psi^{\alpha\beta})}}{B_{3333} + B_{33\rho\sigma}\overline{E_{\rho\sigma}(\psi^{33})}}(\mathbf{x};\tau)\,\mathrm{d}\tau \tag{5.9}$$

$$\phi^{\alpha\beta} = -\xi\psi^{\alpha\beta} + \frac{\partial g^{\alpha\beta}}{\partial\xi}\psi^{33}. \tag{5.10}$$

As usual, the right side of (5.8) must be evaluated at $\eta = \mathbf{x}/\epsilon^a$, $\xi = x_3/\epsilon$.

One computes that (2.23) holds with

$$\begin{aligned} X^{\alpha\beta}_{\lambda\mu} &= -\xi\delta^{\alpha\beta}_{\lambda\mu} + E_{\lambda\mu}(\phi^{\alpha\beta}) \\ X^{\alpha\beta}_{\lambda3} &= 0 \\ X^{\alpha\beta}_{33} &= \frac{\partial g^{\alpha\beta}}{\partial\xi}, \end{aligned} \tag{5.11}$$

where $\delta^{\alpha\beta}_{\lambda\mu}$ is defined by (2.38). Substitution of (5.11) into (2.26) yields a formula for $M_{\alpha\beta\gamma\delta}(\mathbf{x})$; however, a simpler formula may be obtained as follows. Let

$$\begin{aligned} b_{\alpha\beta\gamma\delta}(\mathbf{x};\xi) &= \theta B_{\rho\sigma\lambda\mu}\overline{[\delta^{\alpha\beta}_{\rho\sigma} + E_{\rho\sigma}(\psi^{\alpha\beta})][\delta^{\gamma\delta}_{\lambda\mu} + E_{\lambda\mu}(\psi^{\gamma\delta})]} \\ b_{\alpha\beta33}(\mathbf{x};\xi) &= \theta B_{\lambda\mu33}[\delta^{\alpha\beta}_{\lambda\mu} + \overline{E_{\lambda\mu}(\psi^{\alpha\beta})}] \\ b_{3333}(\mathbf{x};\xi) &= \theta[B_{3333} + B_{\lambda\mu33}\overline{E_{\lambda\mu}(\psi^{33})}]. \end{aligned} \tag{5.12}$$

We see from (5.9) that

$$\frac{\partial g^{\alpha\beta}}{\partial\xi} = \xi b_{\alpha\beta33}/b_{3333}, \tag{5.13}$$

and from (5.10), (5.11) that

$$\begin{aligned} X^{\alpha\beta}_{\lambda\mu} &= -\xi\left[\delta^{\alpha\beta}_{\lambda\mu} + E_{\lambda\mu}(\psi^{\alpha\beta}) - \frac{b_{\alpha\beta33}}{b_{3333}}E_{\lambda\mu}(\psi^{33})\right] \\ X^{\alpha\beta}_{33} &= \xi b_{\alpha\beta33}/b_{3333}. \end{aligned} \tag{5.14}$$

By (5.6), (5.7) and Green's formula,

$$B_{\rho\sigma\lambda\mu}\overline{E_{\rho\sigma}(\psi^{ij})E_{\lambda\mu}(\psi^{kl})} = -B_{\rho\sigma kl}\overline{E_{\rho\sigma}(\psi^{ij})}. \tag{5.15}$$

Substituting (5.14) into (2.26) and using (5.12), (5.15), one obtains

$$M_{\alpha\beta\gamma\delta}(\mathbf{x}) = \int (\xi)^2 \tilde{b}_{\alpha\beta\gamma\delta}(\mathbf{x};\xi)\,\mathrm{d}\xi, \tag{5.16}$$

where the domain of integration is $|\xi| < \max_{\eta} h(\mathbf{x};\boldsymbol{\eta})$, and

$$\tilde{b}_{\alpha\beta\gamma\delta} = b_{\alpha\beta\gamma\delta} - \frac{b_{\alpha\beta 33} b_{\gamma\delta 33}}{b_{3333}}. \tag{5.17}$$

Formulas (5.16) and (5.17) are precisely what one obtains by "homogenizing the plate boundary" first, then using Kirchhoff plate theory. A full explanation of what this means is given in the appendix.

The tensor $M_{\alpha\beta\gamma\delta}$ is once again positive definite: if $t_{\alpha\beta}$ is any 2×2 symmetric tensor and $M_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} = 0$, then $X_{ij}^{\alpha\beta} t_{\alpha\beta} = 0$, by (2.30). It follows from (5.14) that

$$\begin{aligned} b_{\alpha\beta 33} t_{\alpha\beta} &= 0 \\ E_{\lambda\mu}(\psi^{\alpha\beta} t_{\alpha\beta}) &= -t_{\lambda\mu}. \end{aligned} \tag{5.18}$$

Since $Q(\mathbf{x};\xi)$ is connected for sufficiently small $|\xi|$, (5.18) and the periodicity of $\psi^{\alpha\beta}$ imply that $t_{\lambda\mu} = 0$.

If the plate cross-section at height ξ is solid, i.e. if $Q(\mathbf{x};\xi) = \mathbb{R}^2$, then each $\psi^{ij}(\mathbf{x};\boldsymbol{\eta};\xi)$ is independent of $\boldsymbol{\eta}$, and $\tilde{b}_{\alpha\beta\gamma\delta}(\mathbf{x};\xi) = \tilde{B}_{\alpha\beta\gamma\delta}$. These cross-sections make the same contribution in (5.16) as in (2.39); in particular, (5.16) coincides with (2.39) when h is independent of $\boldsymbol{\eta}$. If, on the other hand, the cross-section at height ξ is totally disconnected—in other words, if $Q(\mathbf{x};\xi)$ consists of islands, each contained in a single period cell—then each ψ^{ij} is linear in $\boldsymbol{\eta}$. In this case $b_{\alpha\beta\gamma\delta}(\mathbf{x};\xi) = b_{\alpha\beta 33}(\mathbf{x};\xi) = 0$, and hence $\tilde{b}_{\alpha\beta\gamma\delta}(\mathbf{x};\xi) = 0$. Such cross-sections contribute nothing in (5.16).

6. QUASIPERIODIC STRUCTURE

If $h(\mathbf{x};\boldsymbol{\eta})$ is quasiperiodic in $\boldsymbol{\eta}$, i.e. if (2.13b) holds, then it is in general not possible to construct an asymptotic expansion for $\underline{u}^{\epsilon}$ in powers of ϵ. However, one can obtain the rigidity tensor $M_{\alpha\beta\gamma\delta}(\mathbf{x})$ for this case by a continuity argument based on the periodic one. This procedure is supported by mathematically rigorous analyses of related problems[22, 24].

Given data

$$\begin{aligned} &\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N \in \mathbb{R}^2 \\ &H_0(t_1, \ldots, t_N) \text{ periodic in each } t_i, \end{aligned} \tag{6.1}$$

we shall define a tensor

$$M_{\alpha\beta\gamma\delta} = M_{\alpha\beta\gamma\delta}(H_0; \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N).$$

$M_{\alpha\beta\gamma\delta}$ represents the rigidity tensor of the quasiperiodic plate with rescaled thickness

$$h_0(\boldsymbol{\eta}) = H_0(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\eta}, \ldots, \boldsymbol{\alpha}_N \cdot \boldsymbol{\eta}), \quad \boldsymbol{\eta} = \mathbf{x}/\epsilon^a; \tag{6.2}$$

its definition is different depending on whether $a < 1$, $a = 1$, or $a > 1$. The tensor $M_{\alpha\beta\gamma\delta}(\mathbf{x})$ corresponding to (2.13b) is then obtained by taking

$$H_0(t_1, \ldots, t_N) = H(\mathbf{x}; t_1, \ldots, t_N), \quad \boldsymbol{\alpha}_i = \boldsymbol{\alpha}_i(x), \quad 1 \le i \le N.$$

Suppose first that all the vectors $\boldsymbol{\alpha}_i$ have rational coordinates, and let p be the least common multiple of the denominators; then $h_0(\boldsymbol{\eta})$, defined by (6.2), is periodic with period p in η_1 and η_2. The formulas of Sections 3–5 still apply in this p-periodic context. In particular, (3.2), (4.3)–(4.4), and (5.6)–(5.7) have solutions that are periodic in $\boldsymbol{\eta}$ with period p; and $M_{\alpha\beta\gamma\delta}$ is defined by (3.7), (4.9), or (5.12)–(5.17).

This procedure defines $M_{\alpha\beta\gamma\delta}(H_0; \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$ only for rational vectors $\boldsymbol{\alpha}_i$. The results, however, depend continuously on $\boldsymbol{\alpha}_i$; therefore it makes sense to define

$$M_{\alpha\beta\gamma\delta}(H_0; \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N) = \lim_{\substack{\boldsymbol{\beta}_i \to \boldsymbol{\alpha}_i \\ \boldsymbol{\beta}_i \text{ rational}}} M_{\alpha\beta\gamma\delta}(H_0; \boldsymbol{\beta}_1 \ldots, \boldsymbol{\beta}_N)$$

in the general case.

This approximation procedure may be avoided in $N = 2$ and $\boldsymbol{\alpha}_1 \perp \boldsymbol{\alpha}_2$; that case arises, for example, in modeling a plate with two orthogonal families of stiffeners. One may arrange that $\boldsymbol{\alpha}_1 = (c, 0)$ and $\boldsymbol{\alpha}_2 = (0, d)$ by performing a rotation of the coordinate axes; then the thickness h_0 is periodic, with periods $|c|^{-1}$ and $|d|^{-1}$ in η_1 and η_2 respectively, and so are all the auxiliary functions.

7. EXAMPLE: ONE FAMILY OF STIFFENERS

The different scalings $a < 1$, $a = 1$, and $a > 1$ generally yield very different results for the effective rigidity tensor $M_{\alpha\beta\gamma\delta}$. An instructive example is provided by a single family of rectangular stiffeners, which we model by choosing $h = h(\eta_1)$ as in Fig. 1: h is periodic with period 1, taking two values $h_1 < h_2$, and

$$\mu_i = \text{meas}\{\eta_1 : |\eta_1| < \tfrac{1}{2},\quad h(\eta_1) = h_i\},\quad i = 1, 2.$$

We restrict our attention to the case of an isotropic material, so that Hooke's law is given by (2.8).

For $a < 1$, the case of "slow" thickness variation, the relevant auxiliary functions $\phi^{\alpha\beta}$ are defined by (3.2). One finds that $\phi^{12} = 0$,

$$\frac{\partial^2 \phi^{11}}{\partial \eta_1{}^2} = -1 + c/h^3$$

$$\frac{\partial^2 \phi^{22}}{\partial \eta_1{}^2} = \nu(-1 + c/h^3)$$

where c is the harmonic mean of h^3

$$c = (\mu_1 h_1{}^{-3} + \mu_2 h_2{}^{-3})^{-1}.$$

Denoting by m the arithmetic mean

$$m = \mu_1 h_1{}^3 + \mu_2 h_2{}^3,$$

the nonzero components of $M_{\alpha\beta\gamma\delta}$ for this case are

$$\begin{aligned} &M^{a<1}_{1111} = \frac{2}{3}\frac{E}{1-\nu^2}c \qquad M^{a<1}_{2222} = \frac{2}{3}Em + \frac{2}{3}\frac{E\nu^2}{1-\nu^2}c \\ &M^{a<1}_{1122} = M^{a<1}_{2211} = \frac{2}{3}\frac{E\nu}{1-\nu^2}c \\ &M^{a<1}_{1212} = M^{a<1}_{2112} = M^{a<1}_{1221} = M^{a<1}_{2121} = \frac{E}{3(1+\nu)}m. \end{aligned} \tag{7.1}$$

For $a > 1$, the case of "rapid variation", the auxiliary functions $\boldsymbol{\psi}^{ij}$ solve (5.6), (5.7) on the "slice at height ξ", $Q(\xi)$. If $|\xi| < h_1$ then $\boldsymbol{\psi}^{ij} = 0$; if $h_1 < |\xi| < h_2$, then $\boldsymbol{\psi}^{ij}$ is a linear function of η_1 and the nonzero components of $E_{\alpha\beta}(\boldsymbol{\psi}^{ij})$ are

$$E_{11}(\boldsymbol{\psi}^{11}) = -1 \qquad E_{11}(\boldsymbol{\psi}^{22}) = -\nu/(1-\nu)$$

$$E_{11}(\boldsymbol{\psi}^{33}) = -\nu/(1-\nu) \qquad E_{12}(\boldsymbol{\psi}^{12}) = E_{21}(\boldsymbol{\psi}^{12}) = -\tfrac{1}{2}.$$

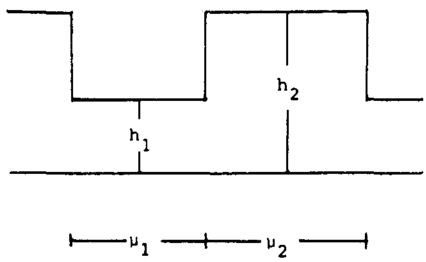

Fig. 1.

The nonzero components of $M_{\alpha\beta\gamma\delta}$ are

$$M^{a>1}_{1111} = \frac{2}{3}\frac{E}{1-\nu^2}h_1{}^3 \qquad M^{a>1}_{2222} = \frac{2}{3}Em + \frac{2}{3}\frac{E\nu^2}{1-\nu^2}h_1{}^3$$
$$M^{a>1}_{1122} = M^{a>1}_{2211} = \frac{2}{3}\frac{E\nu}{1-\nu^2}h_1{}^3 \tag{7.2}$$
$$M^{a>1}_{1212} = M^{a>1}_{2112} = M^{a>1}_{1221} = M^{a>1}_{2121} = \frac{E}{3(1+\nu)}h_1{}^3$$

in this case.

For the intermediate scaling $a = 1$, the auxiliary functions $\underline{\phi}^{11}$, $\underline{\phi}^{22}$, and $\underline{\phi}^{12} = \underline{\phi}^{21}$ are defined by (4.3), (4.4). One verifies that

$$\underline{\phi}^{11} = \frac{E}{1-\nu^2}\underline{\phi}^*, \quad \underline{\phi}^{22} = \frac{E\nu}{1-\nu^2}\underline{\phi}^*,$$

where $\underline{\phi}^* = (\phi_1^*, 0, \phi_3^*)$, and (ϕ_1^*, ϕ_3^*) solves a plane strain elasticity problem on the stiffener cross-section

$$\{(\eta_1, \eta_3): |\eta_1| < \frac{1}{2}, \quad |\eta_3| < h(\eta_1)\}.$$

The function $\underline{\phi}^{12} = (0, \phi_2{}^{12}, 0)$ solves a problem of antiplane shear, which is easily reduced to Laplace's equation for $\phi_2{}^{12}$ on the same two-dimensional domain. By (4.8), (4.9), the dependence of $M_{\alpha\beta\gamma\delta}$ on these auxiliary functions involves only the energies

$$\mathscr{E}^* = \mathscr{M}\left[\int \Sigma_{ij}(\underline{\phi}^*)E_{ij}(\underline{\phi}^*)\,d\eta_3\right]$$
$$\mathscr{E}^{12} = \mathscr{M}\left[\int \Sigma_{ij}(\underline{\phi}^{12})E_{ij}(\underline{\phi}^{12})\,d\eta_3\right],$$

in terms of which

$$M^{a=1}_{1111} = \frac{2}{3}\frac{E}{1-\nu^2}m - \left(\frac{E}{1-\nu^2}\right)^2\mathscr{E}^*$$
$$M^{a=1}_{2222} = \frac{2}{3}\frac{E\nu}{1-\nu^2}m - \left(\frac{E\nu}{1-\nu^2}\right)^2\mathscr{E}^*$$
$$M^{a=1}_{1122} = M^{a=1}_{2211} = \frac{2}{3}\frac{E\nu}{1-\nu^2}m - \frac{E^2\nu}{(1-\nu^2)^2}\mathscr{E}^* \tag{7.3}$$
$$M^{a=1}_{1212} = M^{a=1}_{2112} = M^{a=1}_{1221} = M^{a=1}_{2121} = \frac{1}{3}\frac{E}{1+\nu}m - \mathscr{E}^{12}.$$

We used the FEARS finite element code, developed at the University of Maryland, to compute $\underline{\phi}^*$ and $\underline{\phi}^{12}$ for four choices of h_1, h_2, μ_1, μ_2, using $E = 1.0$ and $\nu = 0.25$. The geometries were selected to have the same total volume. Table 1 gives the computed values of the energies, and Tables 2(a–d) compare the values of $M_{\alpha\beta\gamma\delta}$ corresponding to the scalings $a < 1$, $a = 1$, and $a > 1$.

Table 1. $E = 1.0$, $\nu = 0.25$

	h_1	h_2	μ_1	μ_2	E^*	E^{12}
a)	1/3	2/3	1/2	1/2	.077	.027
b)	1/3	1	3/4	1/4	.149	.062
c)	1/4	3/4	1/2	1/2	.124	.047
d)	1/4	5/4	3/4	1/4	.302	.127

Table 2(a). $h_1 = \frac{1}{3}$, $h_2 = \frac{2}{3}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2}$

	$a < 1$	$a = 1$	$a > 1$
M_{1111}	.047	.031	.026
M_{1122}	.012	.008	.007
M_{2222}	.114	.113	.113
M_{1212}	.044	.017	.010

Table 2(b). $h_1 = \frac{1}{3}$, $h_2 = 1$, $\mu_1 = \frac{3}{4}$, $\mu_2 = \frac{1}{4}$

	$a < 1$	$a = 1$	$a > 1$
M_{1111}	.035	.028	.026
M_{1122}	.009	.007	.007
M_{2222}	.187	.187	.187
M_{1212}	.074	.012	.010

Table 2(c). $h_1 = \frac{1}{4}$, $h_2 = \frac{3}{4}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2}$

	$a < 1$	$a = 1$	$a > 1$
M_{1111}	.021	.014	.011
M_{1122}	.005	.004	.003
M_{2222}	.147	.147	.147
M_{1212}	.058	.011	.004

Table 2(d). $h_1 = \frac{1}{4}$, $h_2 = \frac{5}{4}$, $\mu_1 = \frac{3}{4}$, $\mu_2 = \frac{1}{4}$

	$a < 1$	$a = 1$	$a > 1$
M_{1111}	.015	.012	.011
M_{1122}	.004	.003	.003
M_{2222}	.334	.334	.334
M_{1212}	.133	.006	.004

It is natural to ask which scaling produces the strongest structure. We can show that

$$M^{a>1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \leq M^{a=1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \tag{7.4a}$$

for symmetric tensors $t_{\alpha\beta}$, whenever $h = h(\eta_1)$; in other words, for a single family of stiffeners with given geometry, the $a = 1$ plate is stronger than the $a > 1$ plate. The proof of (7.4a) involves a variational characterization of $M^{a=1}_{\alpha\beta\gamma\delta}$ and $M^{a>1}_{\alpha\beta\gamma\delta}$; it is valid for the general anisotropic Hooke's law (2.5).

Comparing the $a = 1$ and $a < 1$ scalings is more difficult. We conjecture that

$$M^{a=1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \leq M^{a<1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \tag{7.4b}$$

if $h = h(\eta_1)$, for the isotropic Hooke's law (2.8); in other words, we believe that the $a < 1$ plate is the strongest in this context. The data in Tables 2(a–d) support this conjecture. (They are slightly in violation of the condition

$$(M^{a<1}_{1122} - M^{a=1}_{1122})^2 \leq (M^{a<1}_{1111} - M^{a=1}_{1111})(M^{a<1}_{2222} - M^{a=1}_{2222}),$$

but this is due to round-off errors.) Another supporting observation is the inequality

$$M^{a>1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \leq M^{a<1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta}$$

for $h = h(\eta_1)$ as in Fig. 1, which is a consequence of (7.1) and (7.2).

Relation (7.4b) does not extend to the anisotropic case: it is false for some elastic materials and some choices of $h = h(\eta_1)$. We expect (7.4a) to fail as well, for more complicated geometries $h(\eta_1, \eta_2)$. Thus in general the relative strength of the $a < 1$, $a = 1$, and $a > 1$ plates depends on the underlying elastic material, and on the specific form of the thickness variation. These issues will be addressed further in forthcoming article.

REFERENCES

1. S. A. Ambartsumian, *Theory of anisotropic plates. Progress in Materials Science Series*, Vol. 2. Technomic (1970).
2. M. Artola and G. Duvaut, Homogénéisation d'une plaque renforcée. *C.R.A.S.* **284**, 707–710 (1977).
3. M. Artola and G. Duvaut, Homogénéisation d'une plaque renforcée par un système périodique de barres curvilignes. *C.R.A.S.* **286**, 659–662 (1978).
4. I. Babuška and M. Prager, Reissnerian algorithms in the theory of elasticity. *Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys.* **8**, 411–417 (1960).
5. M. P. Bendsøe, Some smear-out models for integrally stiffened plates with applications to optimal design. *Proc. Int. Symp. on Optimum Structural Design*, Tuscon, Arizona, pp. 13.29–13.34.
6. A. Bensoussan, J.-L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam (1978).
7. R. Brizzi and J. P. Chalot, Homogénéisation de frontière. Thèse, Université de Nice (1978).
8. D. Caillerie, Homogénéisation des équations de la diffusion stationnaire dans les domaines cylindriques aplatis. *R.A.I.R.O. Analyse Numérique* **15**, 295–319 (1981).
9. D. Caillerie, Plaques élastiques minces à structure périodique de période et d'épaisseur comparables, *C.R.A.S.* **294**-II, 159–162 (1982).
10. D. Caillerie, Thin elastic and periodic plates, preprint.
11. K.-T. Cheng and N. Olhoff, An investigation concerning optimal design of solid elastic plates. *Int. J. Solids Structures* **17**, 305–323 (1981).
12. K.-T. Cheng, On non-smoothness in optimal design of solid elastic plates. *Int. J. Solids Structures* **17**, 795–810 (1981).
13. K.-T. Cheng and N. Olhoff, Regularized formulation for optimal design of axisymmetric plates. *Int. J. Solids Structures* **18**, 153–169 (1982).
14. P. G. Ciarlet and P. Destuynder, A justification of the two-dimensional linear plate model. *J. Mecanique* **18**, 315–344 (1979).
15. D. Cioranescu and J. Saint Jean Paulin, Homogenization in open sets with holes. *J. Math. Anal. Appl.* **71**, 590–607 (1979).
16. G. Duvaut, Comportement macroscopique d'une plaque perforée périodiquement. In *Singular Perturbations and Boundary Layer Theory*, Lecture Notes in Math. No. 594, pp. 131–145. Springer-Verlag, Berlin (1977).
17. G. Duvaut and A. M. Metellus, Homogénéisation d'une plaque mince en flexion de structure périodique et symetrique. *C.R.A.S.* **283**, 947–950 (1976).
18. B. Fraeijs de Veubeke, *A Course in Elasticity*. Springer-Verlag, Berlin (1979).
19. A. L. Gol'denveizer, Derivation of an approximate theory of bending of a plate by the method of asymptotic integration of the equations of the theory of elasticity. *J. Appl. Math. Mech.* **26**, 1000–1025 (1962).

20. A. L. Gol'denveizer and A. V. Kolos, On the derivation of two-dimensional equations in the thoery of thin plates. *J. Appl. Math. Mech.* **29**, 151–166 (1965).
21. J. N. Goodier, On the problems of the beam and the plate in the theory of elasticity. *Trans. Roy. Soc. Canada* **32**, 65–88 (1938).
22. W. Kohler, G. Papanicolaou and S. Varadhan, Boundary and interface problems in regions with very rough boundaries. In *Multiple Scattering and Waves in Random Media* (Edited by Chow, Kohler and Papanicolaou), pp. 165–170. North-Holland, Amsterdam (1981).
23. R. Kohn and M. Vogelius, A new model for thin plates with rapidly varying thickness II: A convergence proof. *Quart. Appl. Math.*, to appear, 1984.
24. S. M. Kozlov, Averaging differential operators with almost periodic, rapidly oscillating coefficients. *Math. USSR-Sbornik* **35**, 481–498 (1978).
25. W. Kozlowski and Z. Mróz, Optimal design of solid plates. *Int. J. Solids Structures* **5**, 781–794 (1969).
26. T. J. Lardner and E. Reissner, Symmetrical deformations of circular cylindrical shells of rapidly varying thickness. *Donnell 70th Anniversary Volume*, pp. 47–73. University of Houston (1967).
27. F. Lene, Comportement macroscopique de matériaux élastiques comportant des inclusions rigides ou des trous répartis périodiquement. *C.R.A.S.* **286A**, 75–78 (1978).
28. J.-L. Lions, Some mathematical problems connected with solid mechanics. In *Trends in Solid Mechanics* (Edited by J. F. Besseling and A. M. A. van der Heijen). Delft University Press and Sijthoff and Noordhoff (1979).
29. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th Edn Dover (1944).
30. K. A. Lurie and A. V. Cherkaev, *G*-closure of some particular sets of admissible material characteristics for the problem of bending of thin plates. DCAMM Rep. 214, Technical University of Denmark, Lyngby, Denmark (1981).
31. K. A. Lurie and A. V. Cherkaev, Prager theorem application to optimal design of thin plates. *Mech. Solids* **11**(6), 139–141 (1976).
32. K. A. Lurie, A. V. Cherkaev and A. V. Fedorov, Regularization of optimal design problems for bars and plates,—I and II. *J. Opt. Theory Appl.* **37**, 499–543 (1982).
33. G. J. Megarefs, Method of minimal weight design of axisymmetric plates. *Proc. ASCE* **92**, 79–99 (1966).
34. F. Mignot, J. P. Puel and P. M. Suquet, Homogenization and bifurcation of perforated plates. *Int. J. Engng Sci.* **18**, 404–414 (1980).
35. F. Mignot, J. P. Puel and P. M. Suquet, Flambage des plaques elastiques multiperforees. *Ann. Fac. Sci. Toulouse* **III**, 1–57 (1981).
36. D. Morgenstern and I. Szabo, *Vorlesungen über Theoretische Mechanik*. Springer-Verlag, Berlin (1961).
37. D. Morgenstern, Herleitung der plattentheorie aus der dreidimensionalen elastizitätstheorie. *Arch. Rational Mech. Anal.* **4**, 145–152 (1959).
38. R. P. Nordgren, A bound on the error in plate theory. *Quart. Appl. Math.* **28**, 587–595 (1971).
39. N. Olhoff, K. A. Lurie, A. V. Cherkaev and A. V. Fedorov, Sliding regimes and anisotropy in optimal design of vibrating axisymmetric plates. *Int. J. Solids Structures* **17**, 931–948 (1981).
40. E. Reissner, On the derivation of boundary conditions for plate theory. *Proc. Royal Soc., Ser. A*, **276**, 178–186 (1963).
41. E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, No. 127, Springer-Verlag (1980).
42. J. G. Simmonds, An improved estimate for the error in the classical, linear theory of plate bending. *Quart. Appl. Math.* **29**, 439–447 (1971).

APPENDIX

Homogenizing a rough boundary

We explain here the sense in which (5.12)–(5.17) correspond to "homogenizing the rough boundary then using standard plate theory."

The phrase "homogenizing the boundary" refers to the following construction[7]. For $h(\mathbf{x}; \eta)$ periodic in η, let $\mathcal{R}(\epsilon)$ denote the ϵ-dependent domain in $\mathbf{R}^3$

$$\mathcal{R}(\epsilon) = \{\underset{\sim}{x} : |x_3| < h(\mathbf{x}; \mathbf{x}/\epsilon), \mathbf{x} \in \Omega\} \tag{A1}$$

with upper and lower faces $\partial_\pm \mathcal{R}(\epsilon)$. If h is continuous then

$$\partial_\pm \mathcal{R}(\epsilon) = \{\underset{\sim}{x} : x_3 = \pm h(\mathbf{x}; \mathbf{x}/\epsilon), \mathbf{x} \in \Omega\}; \tag{A2}$$

if h has discontinuities then $\partial_\pm \mathcal{R}(\epsilon)$ have vertical regions accordingly. We denote by $\underset{\sim}{u}^\epsilon$ the solution of the elastic equilibrium problem

$$\partial_j[\sigma_{ij}(\underset{\sim}{u}^\epsilon)] = 0 \quad \text{in } \mathcal{R}(\epsilon) \tag{A3}$$

$$\sigma_{ij}(\underset{\sim}{u}^\epsilon) n_j^\epsilon = \begin{cases} 0 & i = 1, 2 \\ f_\pm(\mathbf{x}; \mathbf{x}/\epsilon)|n_3^\epsilon| & i = 3 \end{cases} \text{ on } \partial_\pm \mathcal{R}(\epsilon) \tag{A4}$$

$$\underset{\sim}{u}^\epsilon = 0 \quad \text{on } \{\underset{\sim}{x} : \mathbf{x} \in \partial\Omega, |x_3| < h(\mathbf{x}; \mathbf{x}/\epsilon)\}. \tag{A5}$$

The method of Section 2(d) can be applied to describe $\underset{\sim}{u}^\epsilon$ as $\epsilon \to 0$. We use the same notation as in Section 5. The appropriate ansatz is

$$\underset{\sim}{u}^\epsilon \sim \underset{\sim}{u}^* = \underset{\sim}{w}(\underset{\sim}{x}) + \epsilon \underset{\sim}{\psi}^{ij}(\mathbf{x}; \mathbf{x}/\epsilon; x_3) e_{ij}(\underset{\sim}{w}) \tag{A6}$$

where $\psi^{ij} = (\psi_1{}^{ij}, \psi_2{}^{ij})$ is defined by (5.6), (5.7), and $\psi_3{}^{ij}(\mathbf{x}; \boldsymbol{\eta}; \xi)$ solves

$$\begin{aligned} \frac{\partial}{\partial\eta_\alpha}\left(B_{\alpha3\beta3}\frac{\partial}{\partial\eta_\beta}\psi_3{}^{ij}\right) &= 0 \qquad \text{in } Q(\mathbf{x};\xi) \\ B_{\alpha3\beta3}\frac{\partial\psi_3{}^{ij}}{\partial\eta_\beta}\nu_\alpha &= -B_{\alpha3ij}\nu_\alpha \qquad \text{on } \partial Q(\mathbf{x};\xi). \end{aligned} \tag{A7}$$

By (2.5), only $\psi_3{}^{13}$ and $\psi_3{}^{23}$ are nontrivial.

One computes that

$$e_{ij}(\underline{u}^*) = \mathrm{X}_{ij}^{kl}(\mathbf{x}; \mathbf{x}/\epsilon; x_3)e_{kl}(\underline{w}) + o(\epsilon),$$

with $\mathrm{X}_{ij}^{kl}(\mathbf{x}; \boldsymbol{\eta}; \xi)$ given by

$$\begin{aligned} \mathrm{X}_{\alpha\beta}^{kl} &= \delta_{\alpha\beta}^{kl} + E_{\alpha\beta}(\psi^{kl}) \\ \mathrm{X}_{\alpha3}^{kl} &= \delta_{\alpha3}^{kl} + \frac{1}{2}\frac{\partial\psi_3{}^{kl}}{\partial\eta_\alpha} \\ \mathrm{X}_{33}^{kl} &= \delta_{33}^{kl}. \end{aligned} \tag{A8}$$

The effective equation for $\underline{w}(\underline{x})$ is determined by the variational principle

$$\min_{\underline{w}} \frac{1}{2}\int \mathscr{M}[I_{Q(\mathbf{x};x_3)}B_{rspq}\mathrm{X}_{rs}^{ij}\mathrm{X}_{pq}^{kl}]e_{ij}(\underline{w})e_{kl}(\underline{w})\,\mathrm{d}\underline{x} - \int w_3\cdot\mathscr{F}\,\mathrm{d}\underline{x} \tag{A9}$$

in which $\mathscr{F}(x_1, x_2, x_3)$ is the mean force per unit projected area applied at height x_3; both integrals are over $\mathbf{x}\in\Omega$, $|x_3| < \max_{\boldsymbol{\eta}} h(\mathbf{x}; \boldsymbol{\eta})$; and $\underline{w}$ is constrained by the analogue of (A5). It follows that $\underline{w}$ satisfies the equations of elastostatic equilibrium corresponding to an inhomogeneous "effective medium" with moduli

$$b_{ijkl}(\mathbf{x}; x_3) = \mathscr{M}[I_{Q(\mathbf{x};x_3)}B_{rspq}\mathrm{X}_{rs}^{ij}\mathrm{X}_{pq}^{kl}]. \tag{A10}$$

One verifies that the functions $b_{\alpha\beta\gamma\delta}$, $b_{\alpha\beta33}$, and b_{3333} defined by (A10) are the same as those in Section 5.

The thesis of Brizzi and Chalot[7] shows for a related problem that $u^\epsilon - u^*$ converges to zero in energy, if $h = h(\boldsymbol{\eta})$ does not depend on $\mathbf{x}$ and satisfies certain geometric hypotheses. See also[22] for a related result.

Now we apply Kirchhoff plate theory to an inhomogeneous plate with elastic moduli

$$B_{ijkl}(\underline{x}) = b_{ijkl}(\mathbf{x}; x_3/\epsilon)$$

and slowly-varying thickness

$$|x_3| < \epsilon\cdot\max_{\boldsymbol{\eta}} h(\mathbf{x}; \boldsymbol{\eta}).$$

The ansatz (2.37) must be replaced by

$$\underline{u}^* = (-x_3\partial_1 w, -x_3\partial_2 w, w + \epsilon^2 g^{\alpha\beta}(\mathbf{x}; x_3/\epsilon)\partial_{\alpha\beta}w), \tag{A11}$$

where

$$g^{\alpha\beta}(\mathbf{x}; \xi) = \int_0^\xi \tau\frac{b_{\alpha\beta33}}{b_{3333}}(\mathbf{x}; \tau)\,\mathrm{d}\tau.$$

Repeating the steps of Section 2(e), one is led to formula (5.16).